Near Optimal Broadcast with Network Coding in Large Homogeneous Wireless Networks

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***Abstract*—We propose an efficient broadcast algorithm for wireless sensor networks, based on network coding: we introduce a simple rate selection and analyze its performance (through computation of *min-cut*). By broadcast, we mean sending data from one source to all the other nodes in the network, and our metric for efficiency is the number of transmissions necessary to transmit one packet from the source to every destination.**

We address this problem, in some special cases of wireless "homogeneous" sensor networks contained of the plane: wireless lattice networks, and dense unit disk networks. Our results are based on the simple principle of *Increased Rate for Exceptional Nodes, Identical Rate for Other Nodes* (*IREN/IRON*), for setting rates on the nodes (wireless links) of the network. With this rate selection, we give a value of the maximum broadcast rate of the source: our central result is a proof of the value of the *min-cut* for such networks.

I. Introduction

Seminal work from Ahlswede, Cai, Li and Yeung [1] has introduced the idea of *network coding*, where intermediate nodes are mixing information from different flows (different bits or different packets): one result was that, in the general case, network coding may achieve the maximum information-theoretic capacity for multicast. It is higher, in some cases, than what classical store-and-forward routing could achieve.

One logical domain of application is *wireless sensor networks*, and indeed network coding has been used in wireless networks. In particular, some results include a generalization of the results in [1]: when the loss rates and the capacity of the links are known and fixed, the maximal multicast capacity of the wireless network, can be computed, as shown in [6], [12]. Essentially, for one source, it is the *min-cut* of the network (see section III-A) from the source to the destinations, as for wired networks [1], but considering *hypergraphs* rather than graphs.

However in wireless sensor networks, a primary constraint is not necessarily the capacity of the wireless links: because of the limited battery of each node, the limiting factor is the cost of wireless transmissions. Hence a different focus is *energy-efficiency*, rather than the maximum achievable broadcast rate:

- given one source, minimize the total number of transmissions to achieve the broadcast to destination nodes.

With network coding, the problem turns out to be solvable in polynomial time: for the stated problem, [16], [17] describe methods to find the optimal transmission rate of each node with a linear program. Once the optimal rates are computed,the performance can be asymptotically achieved with *distributed random linear coding* for instance [5], [20]. However, this does not necessarily provide direct insight about the optimal rates, or the optimal cost: those may be obtained by solving the linear program on instances of networks.

Another angle to tackle this problem, would be to explicitly specify the network coding protocol, based on some intuitive foresight, and be able to compute the performance ; for instance [18] starts with exhibiting an optimal algorithm for some simple regular networks.

In general specifying the network coding protocol reduces to specifying the transmission rates for each node [11]. Then the cost is known, and the central element for computing the performance is the estimation of the min-cut of the network. Some results exist about the expected value of the min-cut on some classes of networks: for instance [7] explored the multicast capacity networks where a source which is two hop from the destinations, through a one network of relay nodes ; [21] studied the some classes of unit disk graphs in the plane.

Our approach in a similar spirit. For large-scale sensor networks, one assumption could be that the nodes are distributed in an homogeneous way, and a question would be: "Is there a simple near-optimal rate selection ?" Considering the results of min-cut estimates for random graphs [7], [21], [22], one intuition is that most nodes have similar neighborhood, hence the performance, when setting an identical rate for each node, deserves to be explored. This is the starting point of this paper, and we will focus on homogeneous networks, such as lattice graphs, or random geometric graphs:

1) We introduce a simple rate principle where most nodes have the same transmission rate: IREN/IRON principle (Increased Rate for Exceptional Nodes, Identical Rate for Other Nodes).
2) We give a proof the min-cut for some lattice graphs (modelled as hypergraphs).
3) We deduce an estimate of the min-cut for unit disk hypergraphs.
4) We show that this simple rate selection achieves "near optimal performance", in some classes of homogeneous networks, based on min-cut computation.
5) We illustrate the results obtained by simulations.

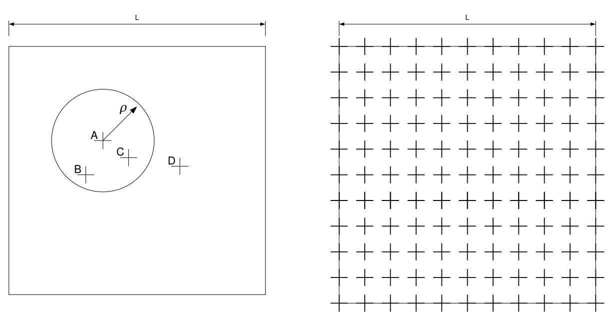
The rest of this paper is organized as follows: section II details the network model and related work; section IV describes the main results (min-cut and near optimality) ; section V gives proofs of min-cut and section VII concludes.

II. Network Model

In this article, we study the problem of broadcasting from one source to all nodes. We will assume an ideal wireless model, infinite capacity: lossless wireless transmissions without collisions or interferences. We also assume that every node has an infinite queue.

Our focus is on large-scale wireless sensor networks. Such networks have been modeled as *unit disk graphs* [27] of the plane, where two nodes are neighbors whenever their distance is lower than a fixed radio range ; see figure 1(a) the principle of unit disk graphs.

An important assumption is that the *wireless broadcast advantage* is used: each transmission is overheard by several nodes. As a result the graph is in reality a *(unit disk) hypergraph*[1]: (it is slightly different from *random geometric graphs* [28] where links are independent). Precisely, the sensor networks considered will be:



(a) Unit disk graph - neighbors of A are B and C since they are within range ρ

(b) Lattice

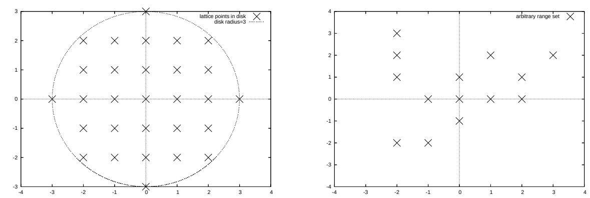
Figure 1. Network Models

- Random unit disk graphs with nodes uniformly distributed (Fig. 1(a))
- Unit disk graphs with nodes organized on a lattice (Fig. 1(b)), special case of the following:
- Lattice sensor networks where the neighborhood of one node is not necessarily the set of nodes within disk like on Fig. 2(a), but may any arbitrary set R such as the one on Fig. 2(b).

Hence for lattice sensor networks, the set R is fixed for one origin node, and all the nodes of the lattice have a similar neighborhood by translation. For simplicity in later proofs, R must include the node itself ($(0,0) \in R$). The following requirement should also be met:

Requirement 1: $\{(-1,0),(1,0),(0,-1),(0,1)\} \subset R$

[1]by abuse of language, the term "unit disk graph" will be used in this article



(a) Disk range for a lattice graph

(b) Arbitrary "range" for a lattice graph

Figure 2. Examples of range for lattices

III. Network Coding Fundamentals

A. Performance of (Wireless) Network Coding : Min-cut

The starting point of network coding is the celebrated work from [1], showing that coding in networks could achieve maximum broadcast capacity (given by the *min-cut*), while in the general case, it is out of reach of traditional transmission methods (i.e. without network coding).

It is possible to model the wireless network as an hypergraph, as done in this section. The benefits of using an hypergraph model, is that it models closely the *wireless broadcast* feature of wireless network, and that there exists a powerful generalization of the results of [1] for network coding for normal graphs. It expresses the maximum broadcast capacity of the graph, when the rates C_v are fixed:

> The maximum broadcast (multicast) rate for a source s to all destinations, is given by the minimum of the *maximum flow capacity* from the source to every individual destination of the network, of the hypergraph [6], [8], [12][2]. This is the *max-flow* which is also to be the *min-cut*.

Of course, this requires the definition of an hypergraph, in section III-A1, and of the max-flow/min-cut, in section III-A2.

1) Hypergraph Notation: Given any set of nodes in a network where the same transmission can reach several neighbors simultaneously, such as with wireless networks, it is possible to describe the connectivity graph as an (oriented) *hypergraph*, following the formalization used in [6], [12] and [8].

An hypergraph is a graph where edges are replaced by *hyperedges*: instead of having one edge linking one head node to one tail node, an hyperedge links one head node to several tail nodes. Precisely,

Following the formalization of [6], [12] and [8], the hypergraph and its min-cut with respect to source s are defined as follows:

• **Hypergraph:** $\mathcal{G} = (\mathcal{V}, \mathcal{H})$, with $\mathcal{V}$ and $\mathcal{H}$ defined as follows:
• **Nodes:** $\mathcal{V} = \{v_i, i = 1, \ldots n\}$, set of vertices (nodes) of the graph (source is $s \in \mathcal{V}$)
• **Hyperedge:** $h_v = (v, H_v)$ where $H_v \subset V$ is the subset of nodes which are reached by one transmission of node v Hence H_v is the set of neighbors of node v.
• **Set of hyperedges:** $\mathcal{H} = \{h_v : v \in \mathcal{V}\}$

[2]actually their results are more general

• **Rate:** Each node v emits on the hyperedge (v, H_v) with a fixed rate C_v.

2) Min-cut of an Hypergraph: Let us consider the source s, and one of the multicast (broadcast) destinations $t \in \mathcal{V}$.

A *cut* is a defined by a partition of the set of vertices V in two sets S, T such as $s \in S$ and $t \in T$. Precisely, because it depends on s and t, it is an *s-t cut*.

Let $Q(s,t)$ the set of the all the *s-t* cuts (S,T).

We denote ΔS, the set of nodes of S for which there is at least one node of T within range. Formally, ΔS is:

$$\Delta S \triangleq \{v \in S : H_v \cap T \neq \emptyset\} \tag{1}$$

The *capacity of the cut* is defined as the maximum rate from the nodes in S to the nodes in T. That is, the capacity of the cut is:

$$C(S) \triangleq \sum_{v \in \Delta S} C_v \tag{2}$$

It is the maximum rate that nodes in the set T taken as a whole, can receive from the nodes in the set S (also taken as a whole). Note that this expression differs from the capacity of a cut when the topology is not an hypergraph, but a graph with simple edges: here, if a node $v \in \Delta S$ can transmit to several nodes of T, its contribution to the capacity is counted only "once", because it is the same transmission (hyperedge), hence same information, that reaches the different nodes.

With this definition of an $s-t$ cut, the $s-t$ *min-cut* is the following:

The *min-cut* between the source s and the destination t is denoted $C_{min}(s,t)$, and is the minimum of the capacity of all the $s-t$ cuts:

$$C_{\min}(s,t) \triangleq \min_{(S,T) \in Q(s,t)} C(S) \tag{3}$$

When multicasting, there are several destinations t for the same source s, hence the min-cut is the minimum of the $s-t$ min-cuts for all t. When broadcasting to all nodes, the min-cut is the minimum for all nodes other than s, and we denote the broadcast min-cut $C_{min}(s)$:

$$C_{\min}(s) \triangleq \min_{t \in \mathcal{V} \setminus \{s\}} C_{\min}(s,t) \tag{4}$$

As indicated in section III-A, $C_{min}(s)$ is the maximum broadcast rate with which the source s can transmit to all the nodes in the network.

B. Related Work

In general specifying the network coding protocol reduces to specifying the transmission rates for each node [11].For minimum-cost multicast, [17] contains several methods (centralized or distributed), to compute the optimal rate selection.However This article is in the spirit of [18] which starts with exhibiting an energy-efficient algorithm for simple networks. The central element for computing the performance is the estimation of the min-cut of the network. We are inspired by the existing techniques and results about the expected value of the min-cut on some classes of networks: for instance [7] explored the multicast capacity networks where a source is two hop from the destinations, through a network of relay nodes ; [21] studied some classes of random geometric graphs. Recently, [22] gave bounds of the min-cut of dual radio networks.

C. Practical Implementation of Wireless Network Coding

It has been shown that a simple form of coding, *linear coding* [2], (using linear combinations of data symbols belonging to Galois fields $\mathcal{F}_p$ - see also [3]), is sufficient to achieve the bounds of [1]. Furthermore, [5] presented one method which does not require coordination of (the coding at) the nodes, by introducing *random linear coding* and by showing that sufficient field size results in high probability of success. With random linear coding, the coding inside the network is no longer predetermined, since it uses random coefficients for the linear combinations.

These works set the path to practical foundations, which are described for instance in [4], [20], and that are used for the simulations given in this article, in section VI.

Vectors: second, the packets are equally sized and are divided into blocks of symbols over a field $\mathcal{F}_p$: content $= (s_1, s_2, ..., s_h)$. As in [4], the packets include a header which is the list of coefficients. Hence the packet format is actually a vector of the format: $(g_1, g_2, \ldots, g_D; s_1, s_2, \ldots, s_h)$.

Transmission: at any point of time, a node of the network has a list of vectors, linear combinations of initial source packets. When the node transmits, it generates a random linear combination of the vectors $v_0, v_1, ..., v_k$ it currently has: $\sum_i \alpha_i v_i$ (where the (α_i) are random coefficients of $\mathcal{F}_p$), and transmit it by wireless broadcasting.

Decoding: once a node has received D linearly independent vectors, it is able to decode the D packets of the generation.

The performance of wireless network coding, when the topology is fixed, and when each node as a fixed rate is know. As shown in [6], it turns out to be the *min-cut* of the wireless network, exactly like for wired networks, except that in this case the wireless network is modelled as an hypergraph.

Similarly the random distributed network coding (see algorithm 1) introduced in [5], can be used, and achieve the maximum given by the *min-cut*.

Moreover, although the algorithm 1 assumes exponential interarrival for the packets, it has been shown that any transmission process with an average rate also achieve optimal rate [8], [11].

Algorithm 1: Random Distributed Network Coding

Nodes' scheduling: Poisson retransmission; the nodes retransmit linear combinations of the vectors that they have, with an exponential interarrival

IV. Our approach: IREN/IRON

A. Overview

As described in the introduction, our approach is to choose an intuitive transmission rate for the each node: essentially, the same rate for most nodes. The rate selection is described in section IV-C. Then, we determine the maximum broadcast rate that can be achieved to transmit from the source to every node in the network as the min-cut of the hypergraph, for both lattice graphs in section IV-D. And finally, from the expression of the cost in section IV-E, we deduce asymptotic optimality (section IV-F).

B. Further Definitions

Consider a network inside a square area such as the one on figure IV-B. We denote L the edge length of square G

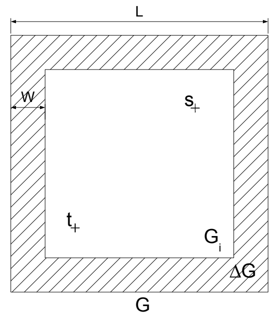


Figure 3. Sample of network inside a square

containing the network. We define the *border area* as the area of fixed width W near the edge of that square, and *border nodes* as the nodes lying in that area. The area $L \times L$ of G is partitioned into:

- ΔG, the border, with area $A_{\Delta G} = 4W(L - W)$ on figure IV-B, the hatched area ΔG
- G_i, the "interior" $G_i \triangleq G \setminus \Delta G$, with area $A_{G_i} = (L - 2W)^2$

Let M be the "expected" number of neighbors of one node. For a lattice network, it is exactly the number of points in the neighborhood R minus one (see Fig. 2(a) and 2(b)): $M = |R| - 1$. For a random disk unit graph with N nodes, the radio range for disk unit graph is denoted ρ. M is related to the density $\mu = \frac{N}{L^2}$ and range as follows: we will take M as the expected number of neighbors $M = \pi\rho^2\mu = \pi\rho^2\frac{N}{L^2}$ of a node which is not in the border area.

One requirement on W is that all nodes in the interior of the square G_i, are out of range of the outside of the network. This is achieved by making W sufficiently large ; for unit disk graph for instance, if the radio range is ρ, then $W = \rho$ satisfies this requirement ; for lattice disk, if R is included in the disk of radius ρ, $W = \rho$ is also a good choice.

C. Rate Allocation with IREN/IRON

The principle IREN/IRON amounts to setting the following transmission rates:

- IREN (Increased Rate for Exceptional Nodes): the rate of transmission is set to M, for the following nodes: the source, and all the border nodes (the "exceptional" nodes).
- IRON (Identical Rate for Other Nodes): every other node, except the source and all the border nodes, transmits with rate 1.

Notice that these rates can be globally scaled by the same amount: the cost and the achieved broadcast rate would linearly increase, and the efficiency would be identical.

1) Rationale for IREN/IRON: .

There are some reasons for the above rate selection. The rationale is the following: we start by imagining an average transmission rate of 1 for mode nodes, the "IRON" part. Then most nodes will receive an average rate of M transmission from their neighbors. With or without network coding, this implies that the maximum achievable broadcast reception rate with this setting is upper-bounded by M.

Now there are two additional issues: the source and the border nodes. For the first, in order to achieve a broadcast reception rate equal to M in the network, the source need to transmit at least with that rate, otherwise it would be a bottleneck.

For the second issue, nodes near the border, one can notice that they have smaller neighborhoods (less than M neighbors). Nevertheless, if they are connected to the network they have at least one neighbor: by setting a rate of M for that neighbor, they are guarantee to received a sufficient rate. Since, in large networks, border nodes represent a minority of nodes, this could have (and does have) limited impact on the efficiency.

After following the steps of the rationale, the main issue is determine whether this insights are sufficient for achieving broadcast rate of M. In this article, we prove that it is the case (see section IV-D) for lattice network, and asymptotically the case for dense unit disk graphs.

However note that this property is not true for general graphs, and that the rate selection hinted here is not absolutely optimal.

D. Performance: Min-Cut (Achievable Broadcast Rate)

The essence of our main result is the following:

Property 1: The min-cut of a lattice graph with the rate selection IREN/IRON is exactly equal to $C_{\min} = M$ (with $M = |R| - 1$).

See section V-A for the proof of this property in Th. 2.

For random unit disk graphs, by mapping the points to an imaginary lattice graph (*embedded lattice*), as an intermediary step, we are able to find bounds of the capacity of random unit disk graphs. This turns out to be much in the spirit of [22].

Precisely we prove the following property:

Property 2: The min-cut $C_{\min}(s)$ of the source s of a random unit disk graph $\mathcal{V}$, is bounded with the min-cut $C^{(\mathcal{L})}_{\min}(s_{\mathcal{L}})$ of some point of the embedded lattice $s_{\mathcal{L}}$ as follows:

$$C_{\min}(s) \geq m_{\min} C^{(\mathcal{L})}_{\min}(s_{\mathcal{L}})$$

where $m_{\min}$ is a random variable related to the number of nodes of the graph mapped to one point of the lattice.

Refer to section V-B3, Th. 3 for details, the property is only quoted here to give this general implication: under some assumptions and definitions, is $m_{\min} C^{(\mathcal{L})}_{\min}(s_{\mathcal{L}})$ actually "close" to M. This is used to deduce an asymptotic result for unit disk graphs:

Property 3: Assume a fixed range. For a sequence of random unit disk graphs $(\mathcal{V}_i)$, with sources s_i, with size $L \to \infty$ and with a density $M \to \infty$ such as $M = \Omega(L^\theta)$, for any fixed $\theta > 0$, we have the following convergence in probability: $\frac{C_{\min}(s)}{M} \xrightarrow{p} 1$
This property is proved in section V-B, Th. 5.

E. Performance: Transmission Cost Per Broadcast

Recall from section I, that the metric for cost is the "number of (packet) transmissions per a (packet) broadcast from the source to the entire network".

The energy cost of broadcasting with IREN/IRON rate selection, can equivalently computed from the rates as the ratio of the number of transmissions per unit time to the number of packets broadcast into the network per unit time. Let us denote E_{cost} this cost per broadcast. Notice that the number of packets broadcast per unit time with (adequate) network coding is the min-cut $C_{\min}(s)$. Then E_{cost} is deduced from the min-cut $C_{\min}$, from the border and interior the areas $A_{\Delta G}, A_{G_i}$, the associated node rates (along with the rates of the nodes of in the border and in the interior) and the node density μ. For fixed W, $M, L \to \infty$:

$$E_{\text{cost}} = \frac{1}{C_{\min}} \mu L^2 \left((1 + O(\frac{1}{L}) + \frac{4MW}{L}(1 + O(\frac{1}{L})) \right)$$

For random unit disk graphs$\mathcal{V}$, E_{cost} is an expected value$E_{\text{cost}} = E(E_{\text{cost}}(\mathcal{V}))$, and $\mu = \frac{N}{L^2}$. For a lattice, $\mu = 1$.

F. Near Optimal Performance for Large Networks

The sections IV-D and IV-E gave the performance and cost with the IREN/IRON principle. As indicated previously, for a given (hyper)graph, the optimal rate selection, and the optimal (minimum-cost) total rate of the network may be computed with a linear program [17]. The optimal cost is not immediately computed and in this section an indirect route is chosen, by using a bound.

Assume that every node has at most $M_{\max}$ neighbors: one single transmission can provide information to $M_{\max}$ nodes at most. Hence in order to broadcast 1 packet to all N nodes, at least $E_{\text{bound}} = \frac{N}{M_{\max}}$ transmissions are necessary.

This is compared to the E_{cost} transmissions per packet broadcast. W.r.t. this bound, let the relative cost be: $E_{\text{rel-cost}} = \frac{E_{\text{cost}}}{E_{\text{bound}}} \geq 1$,

We will prove that $E_{\text{rel-cost}} \to 1$ for some (sequences of) networks:

1) Lattice Graphs.: For lattice graphs, we will assume a constant range, hence a constant neighborhood definition set R, and a constant M, number of neighbors for any node which is not in the border.

The width of the border W is such as, the border includes all nodes that are at distance lower than 2 from the border. Since the size of the neighborhood is kept constant, the width of the border stays also constant. For lattice graphs, W and the neighborhood R are kept fixed (hence also $M = |R| - 1$), whether it is a unit disk lattice graph or not), and only the size L of the network increases to infinity. The number of nodes is $N = L^2$, and $\mu = 1$. The maximum number of neighbors $M_{\max}$ is exactly $M_{max} = M$.

From section IV-E, and from property 1, we have:

$$E_{\text{rel-cost}} = E_{\text{cost}} \frac{M_{\max}}{N} =$$

$$= \left((1 + O(\frac{1}{L}) + \frac{4MW}{L}(1 + O(\frac{1}{L})) \right) = 1 + O(\frac{1}{L})$$

2) Random Unit Disk Graphs.: For random unit disk graphs, first notice that an increase of the density M does not improve the relative cost $E_{\text{rel-cost}}$(due to the cost of border nodes). Now consider a sequence of random graphs, as in property 3, with fixed radio range ρ, fixed border width W, size $L \to \infty$ and with a density $M \to \infty$ such as $M = \Omega(L^\theta)$, for some arbitrary fixed $\theta > 0$, with the additional constraint that $\theta < 1$. We have:

$$E_{\text{rel-cost}} = E_{\text{cost}} \frac{M_{max}}{N} =$$

$$= \frac{M}{C_{\min}} \frac{M_{\max}}{M} \frac{\mu L^2}{N} \left(1 + O(\frac{1}{L}) + \frac{4MW}{L}(1 + O(\frac{1}{L})) \right)$$

Each of part of the product converges towards 1, either surely, or in probability: using property 3, we have the convergence of $\frac{C_{\min}}{M} \xrightarrow{p} 1$, when $L \to \infty$ and similarly with Th. 5 we have $\frac{M_{\max}}{M} \xrightarrow{p} 1$. By definition $N = \mu L^2$. Finally, $M = \Omega(L^\theta)$ for $\theta < 1$ implies that $\frac{4MW}{L} \to 0$.

As a result we have:

$$E_{\text{rel-cost}} \xrightarrow{p} 1$$

in probability, when $L \to \infty$

3) Near Optimality.: The asymptotic optimality is a consequence of the convergence of the cost bound $E_{\text{rel-cost}}$ towards 1. Since it is not possible to have a relative cost $E_{\text{rel-cost}}$ lower than 1, the rate selection IREN/IRON is asymptotically optimal for the two cases presented when $L \to \infty$. Note that, this indirect proof is in fact a stronger statement than optimality of the rate selection in terms of energy-efficiency: it exhibits the fact that asymptotically (nearly) all the transmissions will be *innovative* for the receivers. Note that it is not the case in general, for a given instance of an hypergraph. It evidences the following remarkable fact for the large homogeneous networks considered: network coding may be achieving not only optimal efficiency, but also, asymptotically, perfect efficiency - achieving the information-theoretic bound for each transmission.

V. PROOFS OF THE ACHIEVABLE CAPACITY WITH NETWORK CODING

In this section, we provide a formal proof for both property 1 and property 3 of section IV-D.

A. Proof for Lattice Graphs

1) Overview of the Proof: We first start with a proof for a lattice graph (such as the one Fig. 1(b)). Our objective is to compute prove Th. 2 (section V-A4), which indicates that for one source s, the min-cut $C_{\min}$ of the lattice graph is M (with IREN/IRON).

In order to compute the global min-cut $C_{\min}(s)$, we start with considering one destination node t in the network, and we will provide a bound the min-cut of the (hyper)-graph between s and t, that is, $C_{\min}(s,t)$.

The proof proceeds as follows: we first link the capacity of the cut between nodes in S and nodes in T with the number of nodes in S which are neighbors of nodes in T. The number of these nodes decide the the capacity of the cut. Then we use the fact that the neighbors are obtained with a Minkowski sum. As a result, the inequality on on Minkowski sums could be applied to compute that number of neighbors. However with the effect of the border $\Delta\mathcal{L}$ there are several special cases for applying the inequality, and each time, we prove that the capacity of the cut has the desired bound. The theorem will follow.

2) Preliminaries.: Let Γ be full, unbounded, *integer lattice* in n-dimensional space; it is the set $\mathbb{Z}^n$, where the lattice points are n-tuples of integers.

For lattice graphs, only points on the full lattice are relevant; therefore in this section, the notations $\mathcal{L}, \mathcal{L}_i, \Delta\mathcal{L}$ will be used, for the parts of the full lattice Γ that are in $G, G_i, \Delta G$ respectively.Formally: $\mathcal{L} = \Gamma \cap G, \mathcal{L}_i = \Gamma \cap G_i, \text{and} \Delta\mathcal{L} = \Gamma \cap \Delta G$

The proof is based on the use of the Minkowski addition, and a specific property of discrete geometry (6) below. The Minkowski addition is a classical way to express the neighborhood of one area (for instance, see [25] and the figure 3(a), and figure 4 of that reference).

Given two sets A and B of $\mathbb{R}^n$, the Minkowski sum of the two sets $A \oplus B$ is defined as the set of all vector sums generated by all pairs of points in A and B, respectively:

$$A \oplus B \triangleq \{a + b : a \in A, b \in B\}$$

Consider a subset R of Γ, defining neighborhood, such as the ones on Fig. 2(a) and Fig. 2(b), with origin at the point $(0,0)$. We denote this set R as the *lattice neighborhood definition set*. Then the set of neighbors $\mathcal{N}(t)$ of one node t, with t itself, is:

$$\mathcal{N}(t) \cup \{t\} = \{t\} \oplus R$$

This extends to the neighborhood of a set of points.

The neighbors of t are given with:

$$\mathcal{N}(t) = (\{t\} \oplus R) \setminus \{t\}$$

The rewriting of neighborhood in terms of Minkowski sum, has the advantage that several results of discrete geometry exists, including Brunn-Minkowski-Lysternik type inequalities.

The *Brunn-Minkowski-Lysternik inequality* gives a bound on the size of Minkowski sum of two compact sets of $\mathbb{R}^n$; for integer lattice, there exist several integer variants, including the following one [29]: for two subsets A, B of the integer lattice $\mathbb{Z}^n$,

$$|A \oplus B| \geq |A| + |B| - 1 \tag{5}$$

where $|X|$ represents the number of elements of a subset X of $\mathbb{Z}^n$ For Minkowski sums on the lattice Γ, there exist variants of the *Brunn-Minkowski inequality*, including the following one [29]:

Property 4: For two subsets A, B of the integer lattice $\mathbb{Z}^n$,

$$|A \oplus B| \geq |A| + |B| - 1 \tag{6}$$

where $|X|$ represents the number of elements of a subset X of $\mathbb{Z}^n$

3) Bound on the capacity of one cut $C(S)$.: Consider a lattice $\mathcal{L}$ and a source s. We start with the definition of $C_{\min}(s,t)$ of (3): it requires considering the capacities of every s-t-cut S, T. Let $C(S)$ be the capacity of such a s-t cut $S, T \in Q(s,t)$.

We have the following lemma linking the capacity of the cut and the size of ΔS, the set of nodes of S which are neighbors of nodes of T

Lemma 1: $C(S) \geq |\Delta S|$ (with ΔS defined in (1))

Proof. With the definition in (2), we have:
$C(S) = \sum_{v \in \Delta S} C_v$
$\Rightarrow C(S) \geq \sum_{v \in \Delta S} 1$, because with IREN/IRON, $C_v \geq 1$
$\Rightarrow C(S) \geq |\Delta S|$
which is the lemma. □

Lemma 2: If $U \subset \mathcal{L}_i$ then $U \oplus R \subset \mathcal{L}$

Proof: The requirement on W in section IV-B translates into: for any node $x \in \mathcal{L}_i$, $\{x\} \oplus R \subset \mathcal{L}$, hence the result. ■

Lemma 3: When the requirement V-A3 (in section II) is met, for any two nodes U, V inside the border area, there exist a path using only points for the border area.

Proof: Recall that requirement for the set R which defines the neighborhood (requirement 1 in section **??**) is the following:

> The set R is a subset of Γ and should include the origin point $(0,0)$ as well as the 4 fours points which are immediate neighbors on the lattice: $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$

The requirement is that R should include the 4 immediate neighbors in the directions "left, right, up, and down". Since the border area is a connex area (using this reduced immediate neighborhood definition), the lemma follows. ■

Theorem 1: The capacity of one cut $C(S)$ is such that:

$$C(S) \geq M$$

Proof: There are three possible cases, either the set T has no common nodes with the border $\Delta\mathcal{L}$, or T includes all nodes of $\Delta\mathcal{L}$, or finally T includes only part of nodes in the border area. Formally:

- First case: $T \cap \Delta\mathcal{L} = \emptyset$
- Second case: $\Delta\mathcal{L} \subset T$
- Third case: $T \cap \Delta\mathcal{L} \neq \emptyset$ and $\delta\mathcal{L} \not\subset T$

We will prove inequality of theorem 1 in all 3 cases.

First case, $T \cap \Delta\mathcal{L} = \emptyset$:

With lemma 2, we know that $T \oplus R \subset \mathcal{L}$, hence we can effectively write the neighbors of nodes in T as a Minkowski addition (without getting points in Γ but out of $\mathcal{L}$):

$$\Delta T \triangleq (T \oplus R) \setminus T$$

It follows that:
$|\Delta T| \geq |T \oplus R| - |T|$

Now the inequality (6) can be used:
$|T \oplus R| \geq |T| + |R| - 1$

Hence we get:
$|\Delta T| \geq |T| + |R| - 1 - |T|$, and therefore:

$$|\Delta T| \geq |R| - 1 \quad (7)$$

Recall that S and T form a partition of $\mathcal{L}$; and since ΔT is a subset of $\mathcal{L}$, by definition without any point of of T, we have $\Delta T \subset S$. Hence actually $\Delta T \subset \Delta S$ (with the definition of ΔS in (2)). We can combine this fact with lemma V-A3, lemma 2, and (7), to get:

$|C(S)| \geq |R| - 1$ and the Th. 1 is proved for the first case. □

Second case, $\Delta\mathcal{L} \subset T$:

In this case, all the points of the border area are included in T, and as a consequence, the complementary set of points S has no nodes on the border, i.e. $S \cap \Delta\mathcal{L} = \emptyset$. As a result $S \subset \mathcal{L}_i$.

We will show that a set S has equal or greater number of nodes which are neighbors of nodes in T than $|R| - 1$. The method to prove it is similar with the method of the first case, but we consider neighborhood in the opposite way: we consider the nodes in S that are neighbors of nodes in T.

Let us denote S_i the "interior" of S, that is, the set of nodes of S, which are not in within range of the set T Precisely:
$S_i \triangleq \{x : x \in S \text{ and } (x \oplus R) \cap T = \emptyset\}$

By definition of ΔS in eq. (2), ΔS is the sets of nodes of S which are within range of the set T, and hence the subsets S_i and ΔS form a partition of S

Additionally, because $S_i \subset S$ and $S \subset \mathcal{L}_i$, we know with lemma 2 that $S_i \oplus R \subset \mathcal{L}$. Since by definition of S, $S_i \oplus R$ has no common element with T, and since S and T are a partition of $\mathcal{L}$, the property follows:[3]

$$S_i \oplus R \subset S \quad (8)$$

Now there are two possibilities: either $S_i = \emptyset$ or not.

• If $S_i = \emptyset$, the implication is that $S = \Delta S$, hence in particular, $s \in \Delta S$. Going back to the definition of a cut in (2), we had:
$C(S) = \sum_{v \in \Delta S} C_v$ by definition,
$\Rightarrow C(S) \geq C_s$ because $s \in \Delta S$
$\Rightarrow C(S) \geq M$ because $C_s = M$ with IREN/IRON. and the theorem 1 is proved for the second case, first possibility. □

[3] Alternatively the reader familiar with mathematical morphology [24] could notice that S_i is the *erosion* of S by the structural element R. As a result $S_i \oplus R$ is actually the *opening* of S, and the following property of the opening is known: $S_i \oplus R \subset S$ (see [30] p.40).

• Otherwise, $S_i \neq \emptyset$.

Starting from eq. 8, we had:
$S_i \oplus R \subset S$
$\Rightarrow |S| \geq |S_i \oplus R|$, and as a result, with ineq. 6:

$$|S| \geq |S_i| + |R| - 1 \quad (9)$$

We had established that S_i, ΔS was a partition of S, hence $\Delta S = S \setminus S_i$
$\Rightarrow |\Delta S| \geq |S| - |S_i|$
$\Rightarrow |\Delta S| \geq |R| - 1$

Therefore with lemma , we deduce the capacity of the cut is such that:
$C(S) \geq |R| - 1$
and the theorem 1 is proved for the second case, second possibility. □

Third case: $T \cap \Delta\mathcal{L} \neq \emptyset$ and $\Delta\mathcal{L} \not\subset T$:

Again, since T and S are a partition of $\mathcal{L}$, we deduce that $S \cap \Delta\mathcal{L} \neq \emptyset$; hence both T and S have nodes in the border area $\Delta\mathcal{L}$.

Let us consider such nodes: $u_t \in T \cap \Delta\mathcal{L}$ and $u_s \in S \cap \Delta\mathcal{L}$. With the lemma 2, there exist a path from u_s to u_t with only nodes in the border.

Let us start with u_s, and iterate on the nodes of the path. Since u_s is in S and u_t is in T, we will ultimately find a node of the path u such that u is still in S and that its successor v in the path is not (is in T). By definition of ΔS, $u \in \Delta S$, and also $u \in \Delta\mathcal{L}$ by property of the path.

Hence now, the contribution of u to the capacity of the cut $C(S)$ can be used: $C(S) = \sum_{v \in \Delta S} C_v$ (from def. 2)
$\Rightarrow C(S) \geq C_u$, because $u \in \Delta S$
$\Rightarrow C(S) \geq M$ because $C_u = M$
and the theorem 1 is proved for the third case. □

4) Value of the Min-cut $C_{\min}(s)$: The results of the previous section immediately result in a property on the capacity of every s-t min-cut:

Theorem 2: For any $t \in \mathcal{L}$ different from the source s:

$$C_{\min}(s, t) = M$$

; and as a result: $C_{\min}(s) = M$

Proof: Let S_{min}/T_{min} be one cut with minimal capacity, one such as: $C(S_{min}) = C_{\min}(s, t)$. Applying, the theorem 1, it appears that $C(S_{min}) \geq M$, hence: $C_{\min}(s, t) \geq M$

Conversely let us consider a specific cut, $S_s = \{s\}$ and $T_s = \mathcal{L} \setminus \{s\}$. Obviously s has at least one neighbor, which has to be in T, hence $\Delta S = \{s\}$. The capacity of the cut is $C(S_s) = \sum_{v \in \Delta S} C_v = C_s = M$ and thus $C_{\min}(s, t) \leq M$, and the theorem follows. ■

B. Proof of the Value of Min-Cut for Unit Disk Graphs

In this section, we will prove a probabilistic result on the min-cut, in the case of random unit disk graphs, using an virtual "embedded" lattice. The unit graph will be denoted $\mathcal{V}$, whereas for the embedded lattice the notation of section V is used: $\mathcal{L}$ (along with $\Delta\mathcal{L}$ and $\mathcal{L}_i$). The elements of $\mathcal{V}$ are still called "nodes", but the elements of $\mathcal{L}$ are called "points" to emphasize the fact that they are virtual.

We will assume $W > \rho$ (for instance $W = 2\rho$)

1) Embedded Lattice: Given the square area $L \times L$, we start with fitting a *rescaled* lattice inside it, with a scaling factor r. Precisely, it is the intersection of square G and the set $\{(rx, ry) : (x, y) \in \mathbb{Z}^2\}$.

We will map the points of G to the closest point of the rescaled lattice $\mathcal{L}$: Let us denote $\lambda(x)$, the application which transforms a point u of the Euclidian space $\mathbb{R}^2$ to its closest point of $\mathcal{L}$. Formally, for $u = (x, y) \in \mathbb{Z}^2$,

$$\lambda(x) \triangleq (r\lfloor \frac{x}{r} + \frac{1}{2} \rfloor, r\lfloor \frac{y}{r} + \frac{1}{2} \rfloor)$$

For $u \in \mathcal{L}$, $\lambda^{-1}(u)$ is the set of nodes of $\mathcal{V}$ that are mapped to u. This area of $\mathbb{R}^2$ which is mapped to a same point of the lattice, is a square $r \times r$ around that point. We choose r so that G fits exactly so that such squares are not truncated. This is achieved by taking the origin point of $\mathbb{R}^2$ as the center of the square G, and by selecting $r = \frac{2k+1}{L}$ where k is a positive integer.

Let u be a point of the lattice $\mathcal{L}$, and let denote the $m(u)$ the number of points of $\mathcal{V}$ that are mapped to u with g (they are in the square around u ; and $m(u) \triangleq |\lambda^{-1}(u)|$). Since $\mathcal{V}$ is a random graph, $m(u)$ is a random variable.

Let us denote:

$$m_{\min} \triangleq \min_{u \in \mathcal{L}} m(u) \text{ and } m_{\max} \triangleq \max_{u \in \mathcal{L}} m(u)$$

2) Neighborhood of the Embedded Lattice: We start by defining the neighborhood R for the embedded lattice. The desired property is to have some relationship between neighborhood on the unit graph, and, after mapping, neighborhood on the embedded lattice.

For this, we choose R to be the points of the lattice inside a disk of radius $\rho - 2r$:

$$R(r) = \{(rx, ry) : (rx)^2 + (ry)^2 \leq (\rho - 2r)^2; (x, y) \in \mathbb{Z}^2\}$$

The following lemma shows that we have the desired property.

Lemma 4: Let us consider two nodes of u, v of $\mathcal{V}$ that are mapped on the lattice $\mathcal{L}$ to $u_\mathcal{L}$ and $v_\mathcal{L}$ respectively:

• if $u_\mathcal{L}$ and $v_\mathcal{L}$ are neighbors on the lattice, them u and v are neighbors on the graph $\mathcal{V}$

Proof: We have $\parallel u - v \parallel \leq \parallel u - u_\mathcal{L} \parallel + \parallel u_\mathcal{L} - v_\mathcal{L} \parallel + \parallel v_\mathcal{L} - v \parallel$ using triangle inequality of the Euclidian distance $\parallel . \parallel$.

By definition of neighborhood on the lattice, $u_\mathcal{L} - v_\mathcal{L} \in R(r)$, hence, $\parallel u_\mathcal{L} - v_\mathcal{L} \parallel \leq \rho - 2r$

Moreover since $u_\mathcal{L}$ is the closest point on the lattice of u, and we have $\parallel u - u_\mathcal{L} \parallel \leq \frac{\sqrt{2}}{2} r$ (the length of the half-diagonal of a $r \times r$ square), which is implies $\parallel u - u_\mathcal{L} \parallel \leq r$. The same reasoning applies to v and $v_L at$, and as a result:
$\parallel u - v \parallel \leq 2r + \parallel u_\mathcal{L} - v_\mathcal{L} \parallel \leq \rho$. Hence the lemma. ■

Lemma 5: $|R(r)| \leq \pi \frac{\rho^2}{r^2}$

Proof:

In a similar spirit to the mapping to the lattice, let us consider the square of size $r \times r$ around each point of $R(r)$. Such squares are disjoint for different points of $R(r)$; let us denote $\hat{R(r)}$ the union of all such squares of every point of $R(r)$.

We have, for every point of $u \in \hat{R(r)}$: there exists a point $v \in R(r)$ such that u is in the square around v. Then by a similar argument to lemma 4, $\parallel u - v \parallel \leq \frac{\sqrt{2}}{2} r \leq r$; in addition $\parallel v \parallel \leq \rho - 2r$, from the definition of $R(r)$. Therefore $\parallel u \parallel \leq \rho$, hence $\hat{R(r)}$ is included in the disk of radius ρ. Therefore its area $A(\hat{R(r)})$ verifies $A(\hat{R(r)}) \leq \pi\rho^2$.

In addition, by definition of $\hat{R(r)}$ as union of disjoint squares, we also have another expression of its area: $A(\hat{R(r)}) = |R(r)| r^2$. Using this equality with the previous inequality with $A(\hat{R(r)})$ gives the result. ■

Lemma 6: $|R(r)| = \pi \frac{\rho^2}{r^2} + O(\frac{1}{r})$ when $r \to 0$,

Proof: We can rewrite the definition of $R(r)$ as:

$$R(r) = \{(rx, ry) : x^2 + y^2 \leq (\frac{\rho - 2r}{r})^2; (x, y) \in \mathbb{Z}^2\} \quad (10)$$

It is the number of points in $|R(r)|$ is the number of lattice points within a circle of radius fixed around the origin (the "*circle problem*"). From [30] p. 133, Gauß has shown that $N_c(d) = \pi d^2 + O(d)$, for a circle of radius d, when $d \to \infty$. Here $d = \frac{\rho}{r} - 2$, hence $|R(r)| = \pi(\frac{\rho}{r})^2 + O(\frac{1}{r})$, and the lemma. ■

3) Relationship between Capacities of the Cuts of the Embedded Lattice and the Random Disk Unit Graph.: The idea here is to show that the relationship with a cut of the random unit graph, and a cut of the lattice graph.

Let us consider one source $s \in \mathcal{V}$, one destination $t \in \mathcal{V}$ and the capacity of any S/T cut. Every node of S and T is then mapped to the nearest point of the embedded lattice. For the source, we denote: $s_\mathcal{L} = \lambda(s)$.

An *induced cut* of the embedded lattice is constructed as follows:

- The border area width $W_\mathcal{L}$ is selected so as to be the greatest integer multiple of r which is smaller than W ; and $r < W - \rho$, so that the requirement V-A3 of section II is met.
- For any point of the lattice $v_\mathcal{L} \in \mathcal{L}$, the rate $C^{(\mathcal{L})}_{v_\mathcal{L}}$ is set according to IREN/IRON on the lattice: $C^{(\mathcal{L})}_{v_\mathcal{L}} = |R(r)| - 1$ when $v_\mathcal{L}$ is within the border area of width $W_\mathcal{L}$, and $C^{(\mathcal{L})}_{v_\mathcal{L}} = 1$ otherwise.
- $S_\mathcal{L}$ is the set with the point $s_\mathcal{L}$ and with points of the lattice $\mathcal{L}$, such as only nodes of S are mapped to them:

$$S_\mathcal{L} \triangleq \{s_\mathcal{L}\} \cup \{u_\mathcal{L} : \lambda^{-1}(u_\mathcal{L}) \subset S\} \quad (11)$$

- $T_\mathcal{L}$ is the set of the rest of points of $\mathcal{L}$.

Note that $t \in T_\mathcal{L}$; that all the points of the lattice, to which both points from S and T are mapped, those points are in $T_\mathcal{L}$; and that the points to which no points are mapped are in $S_\mathcal{L}$: $S_\mathcal{L}/T_\mathcal{L}$ is indeed a partition and a $s_\mathcal{L} - t_\mathcal{L}$ cut.

Recall that definition of a cut in eq. 2, we have:
$C(S) = \sum_{v \in \Delta S} C_v$ and $C^{(\mathcal{L})}(S_\mathcal{L}) = \sum_{v \in \Delta S_\mathcal{L}} C_v$ where ΔS and $\Delta S_\mathcal{L}$ are subsets of S and $S_\mathcal{L}$ respectively.

We have the following relationship between these two sets:

Lemma 7: Excluding $s_{\mathcal{L}}$ and s, the nodes of $\mathcal{V}$ that are mapped to points of $\Delta S_{\mathcal{L}}$, are in ΔS ; that is:

$$\lambda^{-1}(\Delta S_{\mathcal{L}} \setminus \{s_{\mathcal{L}}\}) \subset \Delta S \setminus \{s\}$$

Proof:

$\lambda^{-1}(\Delta S_{\mathcal{L}}) = \cup_{u_{\mathcal{L}} \in \Delta S_{\mathcal{L}}} \lambda^{-1}(u_{\mathcal{L}})$ hence it suffice to prove the property for $\lambda^{-1}(u_{\mathcal{L}})$ for every $u_{\mathcal{L}} \in \Delta S_{\mathcal{L}}$.

Let us consider one such point $u_{\mathcal{L}} \in \Delta S_{\mathcal{L}} \setminus \{s_{\mathcal{L}}\}$. By definition of $\Delta S_{\mathcal{L}}$, there exists a point $v_{\mathcal{L}} \in T_{\mathcal{L}}$ within range for $\mathcal{L}$ (that is: $(u_{\mathcal{L}} - v_{\mathcal{L}}) \in R(r)$).

If $\lambda^{-1}(u_{\mathcal{L}}) = \emptyset$, then the property $\lambda^{-1}(u_{\mathcal{L}}) \in \Delta S \setminus \{s\}$ is verified. Hence let us consider the case where $\lambda^{-1}(u_{\mathcal{L}}) \neq \emptyset$:

Since $v_{\mathcal{L}} \in T_{\mathcal{L}}$, by the definition of this set, there exists at least one node of T mapped to $v_{\mathcal{L}}$ and thus: $\lambda^{-1}(v_{\mathcal{L}}) \cap T \neq \emptyset$.

Now consider two points of these non-empty sets, $u \in \lambda^{-1}(u_{\mathcal{L}})$ and $v \in \lambda^{-1}(v_{\mathcal{L}}) \cap T$:

• From lemma 4, we know that u and v are within range ($\| u - v \| \leq \rho$).

• Recall that $\Delta S_{\mathcal{L}} \subset S_{\mathcal{L}}$. By definition of $S_{\mathcal{L}}$, since $u_{\mathcal{L}}$ is in $S_{\mathcal{L}}$, u must be in S.

• $v \in T$

These three conditions imply that $u \in \Delta S$. Also s is mapped to the unique $\lambda(s) = s_{\mathcal{L}}$, therefore $u_{\mathcal{L}} \neq s_{\mathcal{L}}$ implies $u \neq \Delta S \setminus \{s\}$. It follows that $\lambda^{-1}(u_{\mathcal{L}}) \subset \Delta S \setminus \{s\}$, and, as a consequence, the lemma.

It is now possible to use this subset of ΔS to prove the following lemma on relating the cut of $\mathcal{V}$ and its induced cut: ■

Lemma 8: The capacity $C(S)$ of the cut S/T and the capacity of the induced cut $C^{(\mathcal{L})}(S_{\mathcal{L}})$ verify:

$$C(S) \geq m_{\min} C^{(\mathcal{L})}(S_{\mathcal{L}})$$

Proof: First note that if $m_{\min} = 0$, the lemma is proved. Hence, in the rest of the proof, we can assume that this integer verifies $m_{\min} \geq 1$.

In this case, notice that there are $\frac{L^2}{r^2}$ squares of size $r \times r$, each with at least $m_{\min}$ nodes, therefore the total number of nodes verifies: $N \geq \frac{L^2}{r^2} m_{\min}$, and then $\mu \geq \frac{1}{r^2} m_{\min}$ by definition of μ. Combining this with lemma 5, we get:

$$|R(r)| \leq \pi \rho^2 \mu m_{\min} \tag{12}$$

Now consider again the definition of a cut in eq. 2, that can be split in two parts, one without the source, with the source:

$C(S) = \sum_{v \in \Delta S} \ C_v = \sum_{v \in \Delta S \setminus \{s\}} \ C_v + \sum_{v \in \Delta S \cap \{s\}} \ C_v$

With lemma 7, we know a subset of ΔS, hence:

$C(S) \geq C_{\text{interm.}} + C_{\text{src}}$

with $C_{\text{interm.}} = \sum_{v \in \lambda^{-1}(\Delta S_{\mathcal{L}} \setminus \{s_{\mathcal{L}}\})} \ C_v$ and $C_{\text{src}} = \sum_{v \in \Delta S \cap \{s\}} \ C_v$

• The first sum $C_{\text{interm.}}$ can be rewritten as:

$C_{\text{interm.}} = \sum_{v_{\mathcal{L}} \in \Delta S_{\mathcal{L}} \setminus \{s_{\mathcal{L}}\}} \sum_{v \in \lambda^{-1}(v_{\mathcal{L}})} C_v$

Let us consider all the nodes in the square area $\lambda^{-1}(v_{\mathcal{L}})$, and their rates compared to the rate of $v_{\mathcal{L}}$:

- If $C^{(\mathcal{L})}_{v_{\mathcal{L}}} = 1$, then since IREN/IRON assigns only rates ≥ 1, we have $C_v \geq C^{(\mathcal{L})}_{v_{\mathcal{L}}}$ for any $v \in \mathcal{V}$.
- If $C^{(\mathcal{L})}_{v_{\mathcal{L}}} > 1$, $v_{\mathcal{L}}$ is a border node for $\mathcal{L}$ (and $W_{\mathcal{L}}$), and $C^{(\mathcal{L})}_{v_{\mathcal{L}}}$ must actually be $|R(r)| - 1$. Since $W_{\mathcal{L}}$ is chosen so that $W_{\mathcal{L}} < W$, we have also: $\lambda^{-1}(v_{\mathcal{L}})$ is a set of border nodes of $\mathcal{V}$. Their rate is $C_v = \pi \rho^2 M$ by definition.
  From eq. 12, we have $C_v \geq m_{\min} |R(r)|$, hence $C_v \geq |R(r)|$, and finally: $C_v \geq C^{(\mathcal{L})}_{v_{\mathcal{L}}}$

As a result, in both cases, $\forall v \in \lambda^{-1}(v_{\mathcal{L}})$, $C_v \geq C^{(\mathcal{L})}_{v_{\mathcal{L}}}$, and: $\sum_{v \in \lambda^{-1}(v_{\mathcal{L}})} C_{v_{\mathcal{L}}} = |\lambda^{-1}(v_{\mathcal{L}})| C^{(\mathcal{L})}_{v_{\mathcal{L}}} \geq m_{\min} C^{(\mathcal{L})}_{v_{\mathcal{L}}}$

Hence $C_{\text{interm.}}(S) \geq m_{\min} \sum_{v_{\mathcal{L}} \in \Delta S_{\mathcal{L}} \setminus \{s_{\mathcal{L}}\}} C^{(\mathcal{L})}_{v_{\mathcal{L}}}$

• The second sum C_{src} reduces to 0 or 1 term:

- If $s_{\mathcal{L}} \in \Delta S_{\mathcal{L}}$, then $\Delta S_{\mathcal{L}} \cap \{s_{\mathcal{L}}\} = \{s_{\mathcal{L}}\}$.
  With the same reasoning as in the proof of lemma 7, necessarily $s \in \Delta S$ as well, and: $\Delta S \cap \{s\} = \{s\}$.
  $C_{\text{src}} = C_s = \pi \rho^2 \mu$. As before, from eq. 12, we get $C_s \geq m_{\min} |R(r)|$, hence $C_{\text{src}} \geq m_{\min} C^{(\mathcal{L})}_{s_{\mathcal{L}}}$
- If $s_{\mathcal{L}} \notin \Delta S_{\mathcal{L}}$, then $\Delta S_{\mathcal{L}} \cap \{s_{\mathcal{L}}\} = \emptyset$, and obviously $\sum_{v_{\mathcal{L}} \in \Delta S_{\mathcal{L}} \cap \{s_{\mathcal{L}}\}} C^{(\mathcal{L})}_{v_{\mathcal{L}}} = 0$

In both cases, $C_{\text{src}} \geq m_{\min} \sum_{v_{\mathcal{L}} \in \Delta S_{\mathcal{L}} \cap \{s_{\mathcal{L}}\}} C^{(\mathcal{L})}_{v_{\mathcal{L}}}$

Putting together both inequalities for $C_{\text{interm.}}$ and C_{src}, the result is:

$C(S) \geq C_{\text{interm.}} + C_{\text{src}} \geq m_{\min} \sum_{v_{\mathcal{L}} \in \Delta S_{\mathcal{L}} \setminus \{s_{\mathcal{L}}\}} C^{(\mathcal{L})}_{v_{\mathcal{L}}} + m_{\min} \sum_{v_{\mathcal{L}} \in \Delta S_{\mathcal{L}} \cap \{s_{\mathcal{L}}\}} C^{(\mathcal{L})}_{v_{\mathcal{L}}}$

Hence:

$C(S) \geq m_{\min} \sum_{v_{\mathcal{L}} \in \Delta S_{\mathcal{L}}} C^{(\mathcal{L})}_{v_{\mathcal{L}}}$

The right part of the inequality is actually the definition of the capacity of the $s_{\mathcal{L}} - t_{\mathcal{L}}$-cut, hence: $C(S) \geq m_{\min} C^{(\mathcal{L})}(S_{\mathcal{L}})$, which is the lemma. ■

Theorem 3: The min-cut $C_{\min}(s)$ of the graph $\mathcal{V}$, verifies:

$$C_{\min}(s) \geq m_{\min}(|R(r)| - 1)$$

Proof:

From lemma 8, any cut $C(S)$ is lower bounded by $m_{\min} C^{(\mathcal{L})}(S_{\mathcal{L}})$. Since $C^{(\mathcal{L})}(S_{\mathcal{L}})$ is the capacity of a cut of a lattice with IREN/IRON, Th. 2 also indicates that: $C^{(\mathcal{L})}(S_{\mathcal{L}}) \geq C^{(\mathcal{L})}_{\min} = |R(r)| - 1$. Hence the lower bound $m_{\min}(|R(r)| - 1)$ for any $C(S)$, and as a result, for the min-cut $C_{\min}(s)$ ■

4) Nodes of $\mathcal{V}$ Mapped to One Lattice Point.: In Th. 3, $m_{\min}$ plays a central part. In this section, a probabilistic bound is given for the variation of $m_{\min}$.

We start with the following property on random variables: for a variable X which is the sum of n random variables X_i, i.e. $X = \sum_{i=1}^{i=n} X_i$, which are independant and identically distributed, we have the following inequality, which is a Chernoff bound [26]:

$$Pr(X \leq (1 - \delta) E[X]) \leq exp(-\frac{E[X]\delta^2}{2}) \tag{13}$$

for $0 \leq \delta \leq 1$

Symetrically, a similar Chernoff bound exists for the upper tail [26]:

$$Pr(X \leq (1 + \delta) E[X]) \leq exp(-\frac{E[X]\delta^2}{4}) \tag{14}$$

Since $\mathcal{V}$ is a random graph, where points are uniformly distributed, for $u_{\mathcal{L}} \in \mathcal{L}$, the number of points of $\mathcal{V}$ mapped to it, $m(u_{\mathcal{L}})$, is random variable which is the sum of N Bernoulli trials X_v:

$$m(u_{\mathcal{L}}) = \sum_{v \in \mathcal{V}} X_v$$

where X_v is the indicator variable, equal to 1 when v is mapped to $u_{\mathcal{L}}$, and equal to 0 otherwise.

For all v, $E(X_v) = \frac{r^2}{L^2}$, and hence: $E(m(u_{\mathcal{L}})) = \frac{r^2 N}{L^2} = \mu r^2$. The $m(u_i)$ are identically distributed for all $u_i \in \mathcal{L}$. By applying the Chernoff bounds (13) on this sum, we get:

$$Pr[m(u_{\mathcal{L}}) \leq (1-\delta) E[m(u_{\mathcal{L}})]] \leq \exp(-\frac{E[m(u_{\mathcal{L}})]\delta^2}{2}) \quad (15)$$

for $\delta \in]0,1[$.

We can deduce a bound on the probabilities for the minimum $m_{\min}$ of all $m(u)$. For the points $u_i \in \mathcal{L}$, the event $(m_{\min} \leq K)$ implies the event $\big(m(u_1) \leq K$ or $m(u_2) \leq K$ or ... $m(u_{|\mathcal{L}|}) \leq K\big)$:

Hence:

$$Pr[m_{\min} \leq K] \leq Pr[m(u_1) \leq K \texttt{ or } m(u_2) \leq K \texttt{ or } \ldots]$$

Now the different $m(u_i)$ are identically distributed, but are not independent because their sum is exactly N ; but we can use the fact that for two events A and B, $Pr[A \text{ or } B] \leq Pr[A] + Pr[B]$, and then:

$$Pr[m_{\min} \leq K] = \sum_{u \in \mathcal{L}} Pr[m(u) \leq K] = |\mathcal{L}|\, Pr[m(u_1) \leq K]$$

And it follows, with eq. 15:

$$Pr[m_{\min} < (1-\delta)E[m(u)]] \leq |\mathcal{L}| \exp\left(-\frac{E[m(y)]\delta^2}{2}\right)$$

for $\delta \in]0,1[$. Hence, since $|\mathcal{L}| = \frac{L^2}{r^2}$, we have the following theorem 4:

Theorem 4:

$$Pr[m_{\min} \leq (1-\delta)\mu r^2] \leq \exp\left((\log \frac{L^2}{r^2})(1 - \frac{\mu r^2 \delta^2}{2 \log \frac{L^2}{r^2}}) \right)$$

The Th. 4 could be used with Th. 3, to get probabilistic bounds of the min-cut for an instance of a random graph.

Likewise, if we consider the maximum of $m(u)$, $m_{\max} \triangleq \min_{u \in \mathcal{L}} m(u)$, with the upper tail Chernoff bound, the same expression as in Th. 4 is true with $\delta \in]-1,0[$.

5) Asymptotic Values of the Min-Cut of Unit-Disk Graphs.:

Theorem 5: For a sequence of random unit disk graphs and associated source $(\mathcal{V}_i, s_i \in \mathcal{V}_i)$, with fixed radio range ρ, fixed border area width W, with a size $L_i \to \infty$, and a density $M = L^\theta$ with fixed $\theta > 0$, we have the following limit of the min-cut $C_{\min}(s_i)$:

$$\frac{C_{\min}(s_i)}{M} \xrightarrow{p} 1 \text{ in probability. Additionally : } \frac{M_{\max}}{M} \xrightarrow{p} 1$$

Proof: The starting point is Th. 4, which involves several variables: L, μ, δ, and r.The theorem is a result when the size of the network $L \to \infty$ (so that the relative area of the border decreases). We also want:

- $\mu \to \infty$ (that is: $M \to \infty$): the density increases sufficiently fast, so that each square $r \times r$ receives more points and the Chernoff approximation becomes tighter.
- $\delta \to 0$: this ensures $m_{\min}$ converges to its average value as in Th. 4.
- $r \to 0$: in order to have $|R(r)|$ converge to its limit of lemma 6.

By hypothesis, we already have $\mu = \frac{M}{\pi\rho^2} = \frac{1}{\pi\rho^2} L^\theta$ for some $\theta > 0$.

We propose the following settings:

- $\delta = L^{-\frac{\theta}{8}}$; $r = L^{-\frac{\theta}{8}}$

In that case, using Th. 4, we have, for $\delta \in]0,1[$:

$$Pr\left[\frac{m_{\min}}{\mu r^2} \leq (1-\delta)\right] \leq \exp\left((2 - \frac{\theta}{4})(\log L)(1 - \frac{L^{\frac{\theta}{2}}}{(4-\frac{\theta}{2})\log L}) \right)$$

The right side of the inequality converges towards 0 as $L \to \infty$, hence this is a lower bound in probability for $\frac{m_{\min}}{\mu r^2}$.

For the upper bound, notice that $m_{\min}$ is the minimum of the $(m(u_{\mathcal{L}}), u \in \mathcal{L})$, and μr^2 is exactly their average. The minimum cannot be greater than the average hence:

$$Pr[m_{\min} > \mu r^2] = 0$$

,

Hence, we have $\frac{m_{\min}}{\mu r^2} \xrightarrow{p} 1$ in probability, when $L \to \infty$. In a similar way, $\frac{m_{\max}}{\mu r^2} \xrightarrow{p} 1$.

Consider the bound of Th. 3: the min-cut $C_{\min}(s)$ of the graph $\mathcal{V}$, verifies: $C_{\min}(s) \geq m_{\min}(|R(r)| - 1)$, hence:

$$\frac{C_{\min}(s)}{M} \geq \frac{m_{\min}(|R(r)|-1)}{M}$$

The right side of the inequality is:
$a = \frac{m_{\min}(|R(r)|-1)}{M} = \frac{m_{\min}}{\mu r^2} \frac{\mu}{M} r^2 (|R(r)| - 1)$
We have:

- $\frac{m_{\min}}{\mu r^2} \xrightarrow{p} 1$ in probability,
- $\frac{\mu}{M} = \frac{1}{\pi\rho^2}$
- $r^2(|R(r)| - 1) = \pi\rho^2(1 + O(\frac{1}{r}))$, from lemma 6.

Therefore the right side $a \xrightarrow{p} 1$ in probability. This gives an lower bound of $\frac{C_{\min}}{M}$ for $L \to \infty$.

Let us show that this lower bound is also an upper bound (in probability). Recall that the min-cut $C_{\min}$ is lower than any cut, for instance one cut with only neighbors of a node t ($T = \{t\}$). Let us consider the node $t \in \mathcal{V}$ with the maximum number of neighbors $M_{\max}$, and hence $C_{\min} \leq M_{\max}$

We have: the maximum number of neighbors $M_{\max}$ is at most $m_{\max}|R^+(r)|$, where $R^+(r)$ is similar to $R(r)$, except considering squares of around within a point of the lattice with radius $\rho + 2r$. Like for $|R(r)|$, one can prove:

$$R^+(r) = \pi\frac{\rho^2}{r^2}(1 + O(\frac{1}{r}))$$

and like $m_{\min}$, one can show that $\frac{m_{\max}}{\mu r^2} \xrightarrow{p} 1$ in probability.

Collecting these properties, we get:

$$\frac{C_{\min}}{M} \leq \frac{M_{\max}}{M} \leq \frac{m_{\max}|R^{+}(r)|}{M}$$

where the right side of the bounds : is such that $\frac{m_{\max}|R^{+}(r)|}{M} \xrightarrow{p} 1$ in probability.

Hence upper bound, and the theorem. ■

VI. SIMULATIONS

The previous sections have focused on the asymptotic value of the min-cut for large networks. Then random linear network coding can achieve asymptotically the maximum capacity known as the min-cut, when running for an asymptotically infinit time.

In this section, we provide an illustration of the performance of network coding with simulations.

We performed the following types of simulations:

- Performance comparison with store-and-forward bounds: the objective is to show that the performance of broadcasting with network coding with IREN/IRON may outperform what be achieved without network coding (the traditional store-and-forward broadcast), on some examples.
- Min-cut comparison with the average number of neighbors: it illustrates the fact that when broadcasting with IREN/IRON the min-cut approaches the average number of neighbors in wireless networks as the density increases.

A. Comparison with Store-and-Forward

1) Metric for Comparison: For the broadcast of one packet to the entire network, any traditional broadcast method (non-network coding) is characterized by a *Connected Dominating Set* (CDS): it is the set of the nodes which transmitted the packet. Note that the traditional methods need not to explicitly use a such a CDS (like in the case of *MPR-flooding* technique used in [23], which is self-pruning), although several efficient methods do (such as [10]).

To compare network coding and IREN/IRON with traditionnal store-and-forward broadcast, we will proceed following the steps and the logic of [9], as section IV-E also did: the metric for efficiency is the number of transmissions necessary to broadcast one packet to the entire network. In section IV-E, the relative cost $E_{\text{rel}-\text{cost}}$ was the ratio of the total number of transmissions to a bound of the a lower bound number necessary of transmissions $E_{\text{bound}} = \frac{N}{M_{\max}}$. Here, in homogeneous networks, $M_{\max} \approx M$, hence in this section, we will use M instead of $M_{\max}$ in the expression of the bound.

Then the expression of the relative cost $\frac{E_{\text{rel}-\text{cost}}}{E_{\text{bound}}}$, can be reinterpreted as follows: from the point of view of a given node, it is the average ratio of the non-redundant packets received to the number of received packets.

For store-and-forward, "non-redundant packets" means "packets not already received". For network coding, it means "innovative packets" (the ones which that increase the dimension of the vector space of receivers).

We will compare the cost of broadcasting with Network Coding $E^{(\text{nc})}_{\text{rel}-\text{cost}}$ and with the one of any Connected Dominated Set $E^{(\text{cds})}_{\text{rel}-\text{cost}}$. The following notations are used:

NC: $E^{(\text{nc})}_{\text{rel}-\text{cost}} = \frac{T}{G \times \frac{N}{M}}$

CDS: $E^{(\text{cds})}_{\text{rel}-\text{cost}} = \frac{T}{\frac{N}{M}}$

- N: the total number of nodes
- M: the average number of neighbors
- G: the number of packets broadcast (generation size)
- T: the total number of transmissions

With the argument of [9], in any CDS, except for the source, every node must be connected to another node of the CDS: therefore for any common neighbor, the transmission of the second node will be redundant with the transmissions of the first node. A bound on the number of transmission $T^{(\text{cds})}$ can then be computed.

2) Simulation Scenario: In the simulations of this section, we used examples of lattice networks where R (lattice neighborhood definition set) is the four closer neighbors of the lattice).

Precisely $R = \{(0,0), (-1,0), (1,0), (0,-1), (0,1)\}$. The neighborhood of each node fits exactly the minimum requirement 1. This scenario of nodes on a grid with at most four neighbors corresponds to one scenario of [9] (except the lattice considered here is not a torus), and their bound on $E^{(\text{cds})}_{\text{rel}-\text{cost}}$ is $\frac{4}{3}$.

The nodes are on lattice of width $L = 70$ (70×70) and the simulations were performed while increasing the size of *generation* (total number of broadcast packets), and the border width is $W = 2$.

The source s is chosen in the middle of the network.

In general all nodes have same constant transmission rate $\frac{M}{2}$ except the source and nodes which are near the border and have less than M neighbors. The source sends original packets at rate M, and the nodes near the border also send encoded packets at rate M.

For simplicity, the transmissions of nodes in the network are "synchronized", that is, if the transmission rate of one node v is C_v, then the every transmission occurs periodically with a period equal to $\frac{1}{C_v}$

The figure 4 shows the performance of $E^{(\text{nc})}_{\text{rel}-\text{cost}}$ and the bound on $E^{(\text{cds})}_{\text{rel}-\text{cost}}$ with $N = 70 \times 70 = 4900$, $M = 4$ and $G = 20, 40, 60, 80, 100$.

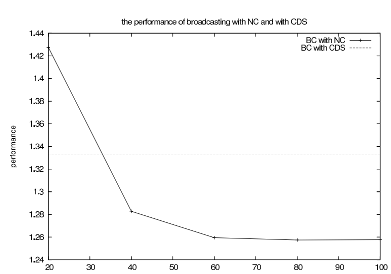


Figure 4. Performance of broadcast with NC and CDS with increasing generation size

As shown in figure 4, the lower bound performance of B_{cds} is constant. (it is the bound of $\frac{4}{3}$)

The performance of B_{nc} becomes better as the generation size increases. The reason that the larger generation size brings the better performance, is the following. At the beginning of the simulations, only the source has new packets, initially only the only transmissions that could bring novel informations are: transmissions from the source, then after that, transmissions from the immediate neighbors of the source, and so on. Hence there is a start-up duration, during which the transmission of nodes further from the source are less likely to bring innovative information to the nodes closer from the source. Similarily, at the end, a similar problem occurs: consider for instance one node which has all the packets from the sources ; then any transmission from a neighbor will bring non-innovative packets. This phenomem explains why efficiency decreases at the end.

This start-up and termination interval durations are independant on the generation size: hence, the efficiency increases together with the size of the generation.

From the figure, we can see confirm that, with our simulations settings, network coding (with IREN/IRON) will outperform any method based on CDS (hence on store and forwards). Notice that [9] established identical results for $M = 4$, but in a scenario where each node had one packet to transmit to every other node. Here we have a single source with several packets to broadcast.

In general, it is not difficult to see that the connectivity constraint gives a lower bound $E^{(\mathrm{cds})}_{\mathrm{rel-cost}} > 1$. For instance, in a unit disk graph, two neighbors share a neighborhood area at least equal to $(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})\rho^2$, hence $E^{(\mathrm{cds})}_{\mathrm{rel-cost}} \geq \frac{6\pi}{2\pi + 3\sqrt{3}}$ with $\frac{6\pi}{2\pi+3\sqrt{3}} \approx 1.6420\ldots$, and as a result, one can expect broadcast with network coding and IREN/IRON to outperform CDS, when the generation size is sufficient, as illustrated by the simulations.

B. Efficiency with Increasing Density in Random Unit Disk Graphs

The previous simulations illustrated the performances on a lattice. For random unit disk graphs, our results have shown that the min-cut, the performance of broadcasting with network coding with IREN/IRON approaches the average number of neighbors in wireless networks as the density increases, that is, with Th. 5, $\frac{C_{\min}}{m} \xrightarrow{p} 1$.

Notice that for a given instance of a random graph, some efficiency is lost when because the min-cut is usually lower than M - unlike for lattices where IREN/IRON results exactly in $C_{\min} = M$.

To give an illustration of this convergence $\frac{C_{\min}}{m} \xrightarrow{p} 1$, we computed the min-cut of random graphs with increasing density. To do so, we modeled oriented hypergraphs as oriented graphs, in the spirit of [16] (refer to the *elementary graphs* and also figure 2 of that reference).

Then, the min-cut was computed from the software library implementing the maxflow algorithm from [32]. The optimizations for tree reuse from [33] were also used.

The network size is $L = 1 \times 1$; the radio range ρ is such that it covers $\frac{1}{25}$ of the network, that is $\rho = \frac{1}{5\sqrt{\pi}} \approx 0.1128\ldots$. We compute the min-cut increasing the network density M, from 125 to 400. As seen in figure 5, the min-cut increases exponentially as the networks become denser and the ratio $\frac{C_{\min}}{M}$ approaches to 1, as expected.

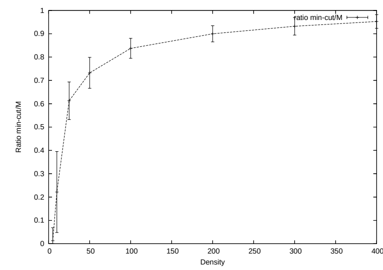


Figure 5. Performance when Increasing Density

VII. Conclusion

We have presented a simple rate selection for network coding for large sensor networks. We computed the broadcast performance from the min-cut with networks modelled as hypergraphs. The central result is that selecting nearly the same rate for all nodes, achieves asymptotic optimality for the "homogeneous" networks that are presented, when the size of the networks becomes larger. This can be translated into the remarkable property: nearly every transmission becomes innovative for the receivers.

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